

K-theory of Leavitt path algebras

P. Ara, M. Brustenga, and G. Cortiñas

Abstract. Let E be a row-finite quiver and let E_0 be the set of vertices of E ; consider the adjacency matrix $N'_E = (n_{ij}) \in \mathbb{Z}^{(E_0 \times E_0)}$, $n_{ij} = \#\{\text{arrows from } i \text{ to } j\}$. Write N_E^t and 1 for the matrices $\in \mathbb{Z}^{(E_0 \times E_0 \setminus \text{Sink}(E))}$ which result from N'_E and from the identity matrix after removing the columns corresponding to sinks. We consider the K -theory of the Leavitt algebra $L_R(E) = L_{\mathbb{Z}}(E) \otimes R$. We show that if R is either a Noetherian regular ring or a stable C^* -algebra, then there is an exact sequence $(n \in \mathbb{Z})$

$$K_n(R)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} K_n(R)^{(E_0)} \longrightarrow K_n(L_R(E)) \longrightarrow K_{n-1}(R)^{(E_0 \setminus \text{Sink}(E))}$$

We also show that for general R , the obstruction for having a sequence as above is measured by twisted nil- K -groups. If we replace K -theory by homotopy algebraic K -theory, the obstructions disappear, and we get, for every ring R , a long exact sequence

$$KH_n(R)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} KH_n(R)^{(E_0)} \longrightarrow KH_n(L_R(E)) \longrightarrow KH_{n-1}(R)^{(E_0 \setminus \text{Sink}(E))}$$

We also compare, for a C^* -algebra \mathfrak{A} , the algebraic K -theory of $L_{\mathfrak{A}}(E)$ with the topological K -theory of the Cuntz-Krieger algebra $C_{\mathfrak{A}}^*(E)$. We show that the map

$$K_n(L_{\mathfrak{A}}(E)) \rightarrow K_n^{\text{top}}(C_{\mathfrak{A}}^*(E))$$

is an isomorphism if \mathfrak{A} is stable and $n \in \mathbb{Z}$, and also if $\mathfrak{A} = \mathbb{C}$, $n \geq 0$, E is finite with no sinks, and $\det(1 - N_E^t) \neq 0$.

1. INTRODUCTION

We consider the K -theory of the Leavitt algebra $L_R(E) = L_{\mathbb{Z}}(E) \otimes R$ of a row-finite quiver E with coefficients in a ring R . To state our results, we need some notation. Let E_0 be the set of vertices of E ; consider the adjacency matrix $N'_E = (n_{ij}) \in \mathbb{Z}^{(E_0 \times E_0)}$, $n_{ij} = \#\{\text{arrows from } i \text{ to } j\}$. Write N_E^t and 1 for the matrices $\in \mathbb{Z}^{(E_0 \times E_0 \setminus \text{Sink}(E))}$ which result from N'_E and from the identity matrix after removing the columns corresponding to sinks. Our results relate the K -theory of $L_R(E)$ with the spectrum

$$C = \text{hocofiber}(K(R)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} K(R)^{(E_0)})$$

In terms of homotopy groups, the fundamental property of C is that there is a long exact sequence $(n \in \mathbb{Z})$

$$(1.1) \quad K_n(R)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} K_n(R)^{(E_0)} \longrightarrow \pi_n(C) \longrightarrow K_{n-1}(R)^{(E_0 \setminus \text{Sink}(E))}$$

For a rather general class of rings (which includes all unital ones) and all row-finite quivers E , we show (Theorem 6.3) that there is a naturally split injective map

$$(1.2) \quad \pi_*(C) \rightarrow K_*(L_R(E))$$

The cokernel of (1.2) can be described in terms of twisted nil- K -groups (see 5.10, 6.6). We show that these nil- K -groups vanish for some classes of rings R , including the following two cases:

- R is a regular supercoherent ring (see 7.6). In particular this covers the case where R is a Noetherian regular ring.
- R is a stable C^* -algebra (see 9.12).

Received by the editors March 3, 2009.

The first and second named authors were partially supported by DGI MICIIN-FEDER MTM2008-06201-C02-01, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The third named author was supported by CONICET and partially supported by grants PICT 2006-00836, UBACyT X051, and MTM2007-64074.

In particular for such R we get a long exact sequence

$$(1.3) \quad K_n(R)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} K_n(R)^{(E_0)} \rightarrow K_n(L_R(E)) \rightarrow K_{n-1}(R)^{(E_0 \setminus \text{Sink}(E))}$$

We also consider Weibel's homotopy algebraic K -theory $KH_*(L_R(E))$. We show in 8.6 that for any ring R and any row-finite quiver, there is a long exact sequence

$$KH_n(R)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} KH_n(R)^{(E_0)} \rightarrow KH_n(L_R(E)) \rightarrow KH_{n-1}(R)^{(E_0 \setminus \text{Sink}(E))}$$

There is a natural comparison map $K_* \rightarrow KH_*$; if R is a regular supercoherent ring or a stable C^* -algebra, then $K_*(R) \rightarrow KH_*(R)$ and $K_*(L_R(E)) \rightarrow KH_*(L_R(E))$ are isomorphisms, so the sequences agree in these cases. We further compare, for a C^* -algebra \mathfrak{A} , the algebraic K -theory of $L_{\mathfrak{A}}(E)$ with the topological K -theory of the Cuntz-Krieger algebra $C_{\mathfrak{A}}^*(E)$; we show that the natural map

$$\gamma_n^{\mathfrak{A}}(E) : K_n(L_{\mathfrak{A}}(E)) \rightarrow K_n(C_{\mathfrak{A}}^*(E)) \rightarrow K_n^{\text{top}}(C_{\mathfrak{A}}^*(E))$$

is an isomorphism in some cases, including the following two:

- $\mathfrak{A} = \mathbb{C}$, E is finite with no sinks, $\det(1 - N_E^t) \neq 0$, and $n \geq 0$ (see 9.4).
- \mathfrak{A} is stable, E is row-finite, and $n \in \mathbb{Z}$ (see 9.13).

The rest of this paper is organized as follows. In Section 2 we recall the results of Suslin and Wodzicki on excision in K -theory and draw some consequences which are used further on in the article. The most general result on excision in K -theory, due to Suslin [26], characterizes those rings A on which K -theory satisfies excision in terms of the vanishing of Tor groups over the unitalization $\tilde{A} = A \oplus \mathbb{Z}$. Namely A satisfies excision if and only if

$$(1.4) \quad \text{Tor}_*^{\tilde{A}}(\mathbb{Z}, A) = 0 \quad (* \geq 0).$$

We call a ring A H' -unital if it satisfies (1.4); if A is torsion-free as an abelian group, this is the same as saying that R is H -unital in the sense of Wodzicki [33]. We show in Proposition 2.8 that if A is H' -unital and $\phi : A \rightarrow A$ is an automorphism, then the same is true of both the twisted polynomial ring $A[t, \phi]$ and the twisted Laurent polynomial ring $A[t, t^{-1}, \phi]$. We recall that, for unital A , the K -theory of the twisted Laurent polynomials was computed in [18] and [34]. If R is a unital ring and $\phi : R \rightarrow pRp$ is a corner isomorphism, the twisted Laurent polynomial ring is not defined, but the corresponding object is the corner skew Laurent polynomial ring $R[t_+, t_-, \phi]$ of [3]. In Section 3 we use the results of [34] and of Section 2 to compute the K -theory of $R \otimes A[t_+, t_-, \phi \otimes 1]$ for (R, ϕ) as above, and A any nonunital algebra such that $R \otimes A$ is H' -unital (Theorem 3.6). In the next section we consider the relation between two possible ways of defining the incidence matrix of a finite quiver, and show that the sequences of the form (1.1) obtained with either of them are essentially equivalent (Proposition 4.4). In Section 5 we use the results of the previous sections to compute the K -theory of the Leavitt algebra of a finite quiver with no sources with coefficients in an H' -unital ring (Theorem 5.10). The general case of row-finite quivers is the subject of Section 6. Our most general result is Theorem 6.3, where the existence of the split injective map (1.2) is proved for the Leavitt algebra $L_A(E)$ of a row-finite quiver E . In the latter theorem, A is required to be either a ring with local units, or a \mathbb{Z} -torsion free H' -unital ring. In Section 7 we specialize to the case of Leavitt algebras with regular supercoherent coefficient rings. We show that the sequence (1.3) holds whenever R is regular supercoherent (Theorem 7.6). For example this holds if R is a field, since fields are regular supercoherent; this particular case, for finite E , is used in [2] to compute the K -theory of the algebra $Q_R(E)$ obtained from $L_R(E)$ after inverting all square matrices with coefficients in the path algebra $P_R(E)$ which are sent to invertible matrices by the augmentation map $P_R(E) \rightarrow R^{E_0}$. Section 8 is devoted to homotopy algebraic K -theory, KH . For a unital ring R , a corner isomorphism $\phi : R \rightarrow pRp$, and a ring A , we compute the KH -theory of $R \otimes A[t_+, t_-, \phi \otimes 1]$ (Theorem 8.4). Then we use this to establish the sequence (??) for any row finite quiver E and any coefficient ring A (Theorem 8.6). In the last section we compare the K -theory of the Leavitt algebra $L_{\mathfrak{A}}(E)$ with coefficients in a C^* -algebra \mathfrak{A} with the topological K -theory of the corresponding Cuntz-Krieger algebra $C_{\mathfrak{A}}^*(E)$. In Theorem 9.1 we establish the spectrum-level version of the well-known calculation of the topological K -theory of the Cuntz-Krieger algebra $C_{\mathfrak{A}}^*(E)$ of a row-finite quiver E with coefficients in a C^* -algebra \mathfrak{A} . Theorem 9.4 shows that if E is a finite quiver without sinks and such that $\det(1 - N_E^t) \neq 0$, then the natural map $\gamma_n^{\mathfrak{A}} : K_n(L_{\mathfrak{A}}(E)) \rightarrow K_n^{\text{top}}(C_{\mathfrak{A}}^*(E))$ is an isomorphism for $n \geq 0$ and the

zero map for $n \leq -1$. In Theorem 9.13 we show that if \mathfrak{B} is a stable C^* -algebra, then $\gamma_n^{\mathfrak{B}}$ is an isomorphism for all $n \in \mathbb{Z}$.

2. H' -UNITAL RINGS AND SKEW POLYNOMIAL EXTENSIONS

Let R be a ring and $\tilde{R} = R \oplus \mathbb{Z}$ its unitization. We say that R is H' -unital if

$$\mathrm{Tor}_*^{\tilde{R}}(R, \mathbb{Z}) = 0 \quad (* \geq 0).$$

Note that, for any, not necessarily H' -unital ring R ,

$$\mathrm{Tor}_*^{\tilde{R}}(\mathbb{Z}, R) = \mathrm{Tor}_{*+1}^{\tilde{R}}(\mathbb{Z}, \mathbb{Z}) = \mathrm{Tor}_*^{\tilde{R}}(R, \mathbb{Z}) \quad (* \geq 0)$$

Thus all these Tor groups vanish when R is H' -unital; moreover, in that case we also have

$$\mathrm{Tor}_*^{\tilde{R}}(R, R) = 0 \quad (* \geq 1), \quad \mathrm{Tor}_0^{\tilde{R}}(R, R) = R^2 = R.$$

A right module M over a ring R is called H' -unitary if $\mathrm{Tor}_*^{\tilde{R}}(M, \mathbb{Z}) = 0$. The definition of H' -unitary for left modules is the obvious one.

Example 2.1. If R is H' -unital then it is both right and left H' -unitary as a module over itself. Let $\phi : R \rightarrow R$ be an endomorphism. Consider the bimodule ${}_R\phi R$ with left multiplication given by $a \cdot x = \phi(a)x$ and the usual right multiplication. As a right module, ${}_R\phi R \cong R$, whence it is right H' -unitary. If moreover ϕ is an isomorphism, then it is also isomorphic to R as a left module, via ϕ , and is thus left H' -unitary too.

Remark 2.2. The notion of H' -unitarity is a close relative of the notion of H -unitarity introduced by Wodzicki in [33]. The latter notion depends on a functorial complex $C^{\mathrm{bar}}(A)$, the *bar complex* of A ; we have $C_n^{\mathrm{bar}}(A) = A^{\otimes n+1}$. The ring A is called H -unital if for all abelian groups V , the complex $C^{\mathrm{bar}}(A) \otimes V$ is acyclic. If A is flat as a \mathbb{Z} -module, then $C^{\mathrm{bar}}(A)$ is a complex of flat \mathbb{Z} -modules and $H_*(C^{\mathrm{bar}}(A)) = \mathrm{Tor}_*^{\tilde{A}}(\mathbb{Z}, A)$. Hence H' -unitarity is the same as H -unitarity for rings which are flat as \mathbb{Z} -modules. Unital rings are both H and H' -unital. Because C^{bar} commutes with filtering colimits, the class of H -unital rings is closed under such colimits. Similarly, there is also a functorial complex which computes $\mathrm{Tor}_*^{\tilde{A}}(\mathbb{Z}, A)$ and which commutes with filtering colimits ([7, 6.4.3]); hence also the class of H' -unital rings is closed under filtering colimits. If A is H or H' -unital then the same is true of the matrix ring $M_n A$. In the H -unital case, this is proved in [33, 9.8]; the H' -unital case follows from a theorem of Suslin cited below (Theorem 2.6). The class of H -unital rings is furthermore closed under tensor products ([28, 7.10]).

Lemma 2.3. *Let A be a ring. If A is H' -unital, then $A \otimes \mathbb{Q}$ is H' -unital.*

Proof. Tensoring with \mathbb{Q} over \mathbb{Z} is an exact functor from \tilde{A} -modules to $\tilde{A} \otimes \mathbb{Q}$ -modules which preserves free modules. Hence if $L \rightarrow A$ is a free \tilde{A} -resolution, then $L \otimes \mathbb{Q} \rightarrow A \otimes \mathbb{Q}$ is a free $\tilde{A} \otimes \mathbb{Q}$ -resolution. Moreover,

$$\mathbb{Q} \otimes_{\tilde{A} \otimes \mathbb{Q}} L \otimes \mathbb{Q} = L \otimes \mathbb{Q}/A \cdot L \otimes \mathbb{Q} = (L/A \cdot L) \otimes \mathbb{Q}$$

Hence

$$\mathrm{Tor}_*^{\tilde{A} \otimes \mathbb{Q}}(A \otimes \mathbb{Q}, \mathbb{Q}) = \mathrm{Tor}_*^{\tilde{A}}(\mathbb{Z}, A) \otimes \mathbb{Q}$$

Thus A H' -unital implies that $0 = \mathrm{Tor}_*^{\tilde{A} \otimes \mathbb{Q}}(A \otimes \mathbb{Q}, \mathbb{Q})$. But by [33, §2], $\mathrm{Tor}_*^{\tilde{A} \otimes \mathbb{Q}}(A \otimes \mathbb{Q}, \mathbb{Q}) = H_*(C^{\mathrm{bar}}(A \otimes \mathbb{Q}))$. Thus $A \otimes \mathbb{Q}$ is H -unital, and therefore H' -unital. \square

Corollary 2.4. *If A and B are H' -unital, and B is a \mathbb{Q} -algebra, then $A \otimes B$ is H' -unital.*

Proof. It follows from the previous lemma and from the fact (proved in [28, 7.10]) that the tensor product of H -unital \mathbb{Q} -algebras is H -unital. \square

Example 2.5. The basic examples of H' -unital rings we shall be concerned with are unital rings and C^* -algebras. The fact that the latter are H' -unital follows from the results of [26] and [28] (see [7, 6.5.2] and Theorem 2.6 below). If A is an H' -unital ring and \mathfrak{B} a C^* -algebra, then $A \otimes \mathfrak{B}$ is H' -unital, by Corollary 2.4.

A ring R is said to *satisfy excision* in K -theory if for every embedding $R \triangleleft S$ of R as a two-sided ideal of a unital ring S , the map $K(R) = K(\tilde{R} : R) \rightarrow K(S : R)$ is an equivalence. One can show (see e.g. [6, 1.3]) that if R satisfies excision in K -theory and R is an ideal in a nonunital ring T , then the map $K(R) \rightarrow K(T : R)$ is an equivalence too.

The main result about H' -unital rings which we shall need is the following.

Theorem 2.6. ([26]) *A ring R is H' -unital if and only if it satisfies excision in K -theory.*

Using the theorem above we get the following Morita invariance result for H' -unital rings.

Lemma 2.7. *Let R be a unital ring, $e \in R$ an idempotent. Assume e is full, that is, assume $ReR = R$. Further let A be a ring such that both $R \otimes A$ and $eRe \otimes A$ are H' -unital. Then the inclusion map $eRe \otimes A \rightarrow R \otimes A$ induces an equivalence $K(eRe \otimes A) \rightarrow K(R \otimes A)$.*

Proof. Put $S = R \otimes \tilde{A}$, and consider the idempotent $f = e \otimes 1 \in S$. One checks that f is a full idempotent, so that $K(fSf) \rightarrow K(S)$ is an equivalence. Now apply excision. \square

Proposition 2.8. *Let R be a ring and $\phi : R \rightarrow R$ an automorphism. Assume R is H' -unital. Then $R[t, \phi]$ and $R[t, t^{-1}, \phi]$ are H' -unital rings.*

Proof. If P is a projective resolution of R as a right \tilde{R} -module, then $P \otimes_{\tilde{R}} \widetilde{R[t, \phi]}$ is a complex of right $\widetilde{R[t, \phi]}$ -projective modules. Moreover, we have an isomorphism of R -bimodules

$$\widetilde{R[t, \phi]} = \tilde{R} \oplus \bigoplus_{n=1}^{\infty} Rt^n \cong \tilde{R} \oplus \bigoplus_{n=1}^{\infty} R_{\phi^n}$$

Thus, because R is assumed H' -unital,

$$H_*(P \otimes_{\tilde{R}} R) = \text{Tor}_*^{\tilde{R}}(R, R) = \begin{cases} 0 & * \geq 1 \\ R & * = 0 \end{cases}$$

Here we have used only the left module structure of R ; the identities above are compatible with any right module structure, and in particular with both the usual one and that induced by ϕ^n . It follows that

$$Q = P \otimes_{\tilde{R}} \widetilde{R[t, \phi]}$$

is a projective resolution of $R[t, \phi]$ as a right $\widetilde{R[t, \phi]}$ -module. Since $\tilde{R} \rightarrow \widetilde{R[t, \phi]}$ is compatible with augmentations, we have

$$Q \otimes_{\widetilde{R[t, \phi]}} \mathbb{Z} = P \otimes_{\tilde{R}} \mathbb{Z}$$

Hence $R[t, \phi]$ is H' -unital. Next we consider the case of the skew Laurent polynomials. We have a bimodule isomorphism

$$\widetilde{R[t, t^{-1}, \phi]} = \tilde{R} \oplus \bigoplus_{n=1}^{\infty} (Rt^n \oplus t^{-n}R) \cong \tilde{R} \oplus \bigoplus_{n=1}^{\infty} (R_{\phi^n} \oplus {}_{\phi^n}R)$$

Thus since ${}_{\phi^n}R$ is left H' -unitary, the same argument as above shows that $R[t, t^{-1}, \phi]$ is H' -unital. \square

3. K -THEORY OF TWISTED LAURENT POLYNOMIALS

Let X , N_+ , N_- and Z be objects in a triangulated category \mathcal{T} . Let $\phi : X \rightarrow X$ and $j^\pm : X \oplus N_\pm \rightarrow Z$ be maps in \mathcal{T} . Let $i^\pm : X \rightarrow X \oplus N_\pm$ be the inclusion maps. Define a map

$$\psi = \begin{bmatrix} i^+ & i^+ \\ i^- & i^- \circ \phi \end{bmatrix} : X \oplus X \rightarrow (X \oplus N_+) \oplus (X \oplus N_-),$$

Lemma 3.1. *Assume*

$$X \oplus X \xrightarrow{\psi} (X \oplus N_+) \oplus (X \oplus N_-) \xrightarrow{[j^+, j^-]} Z \xrightarrow{\partial} \Sigma X \oplus \Sigma X$$

is an exact triangle in \mathcal{T} . Then

$$(3.2) \quad X \xrightarrow{[0, 1 - \phi, 0]} N_+ \oplus X \oplus N_- \xrightarrow{[j^+|_{N_+}, -j^-|_X, j^-|_{N_-}]} Z \xrightarrow{\partial'} \Sigma X$$

is an exact triangle in \mathcal{T} , for suitable ∂' . In particular,

$$Z \cong N_+ \oplus N_- \oplus \text{cone}(1 - \phi : X \rightarrow X).$$

Proof. Note that

$$\psi = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & \phi \\ 0 & 0 \end{bmatrix}.$$

Consider the maps

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \psi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 - \phi \\ 0 & 0 \end{bmatrix}, \text{ and } \psi_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Write $\psi = \psi_1 \psi_2 \psi_3$. Then we have an exact triangle

$$X \oplus X \xrightarrow{\psi_2} (X \oplus N_+) \oplus (X \oplus N_-) \xrightarrow{\tilde{j}} Z \xrightarrow{\partial''} \Sigma X \oplus \Sigma X$$

Here $\tilde{j} = [j^+, j^-]\psi_1$ and $\partial'' = \psi_3 \partial$. The result follows. \square

Let $\phi : X \rightarrow X$ be a map of spectra. We write $\phi^{-1}X$ for the colimit of the following direct system

$$X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} \dots$$

Lemma 3.3. *Let X and ϕ be as above, and consider the map $\hat{\phi} : \phi^{-1}X \rightarrow \phi^{-1}X$ induced by ϕ . Then*

$$\text{hocofiber}(1 - \phi : X \rightarrow X) \cong \text{hocofiber}(1 - \hat{\phi} : \phi^{-1}X \rightarrow \phi^{-1}X)$$

Proof. Write $\hat{\phi} : \phi^{-1}X \rightarrow \phi^{-1}X$ for the induced map; we have a homotopy commutative diagram

$$(3.4) \quad \begin{array}{ccccc} \phi^{-1}X & \xrightarrow{1 - \hat{\phi}} & \phi^{-1}X & \longrightarrow & \text{hocofiber}(1 - \hat{\phi}) \\ \uparrow & & \uparrow & & f \uparrow \\ X & \xrightarrow{1 - \phi} & X & \longrightarrow & \text{hocofiber}(1 - \phi) \end{array}$$

Both the top and bottom rows are fibration sequences. We have to show that the map of stable homotopy groups $f_n : \pi_n \text{hocofiber}(1 - \phi) \rightarrow \pi_n \text{hocofiber}(1 - \hat{\phi})$ induced by f is an isomorphism. Denote by ϕ_n the endomorphism of $\pi_n(X)$ induced by ϕ . Note that ϕ_n induces a $\mathbb{Z}[t]$ -action on $\pi_n X$, and that

$$\pi_n(\phi^{-1}X) = \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]} \pi_n X =: \phi_n^{-1} \pi_n X$$

It follows that the long exact sequence of homotopy groups associated to the top fibration of (3.4) is the result of applying the functor $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]}$ to that of the bottom. In particular the left and right vertical maps in the diagram below are isomorphisms

$$\begin{array}{ccccccc} 0 \rightarrow \text{coker}(1 - \hat{\phi}_n) & \longrightarrow & \pi_n(\text{hocofiber}(1 - \hat{\phi})) & \longrightarrow & \ker(1 - \hat{\phi}_{n+1}) & \rightarrow 0 \\ \uparrow & & f_n \uparrow & & \uparrow & & \\ 0 \rightarrow \text{coker}(1 - \phi_n) & \longrightarrow & \pi_n(\text{hocofiber}(1 - \hat{\phi})) & \longrightarrow & \ker(1 - \phi_{n+1}) & \rightarrow 0 \end{array}$$

It follows that f is an equivalence, as wanted. \square

It will be useful to introduce the following notation.

Notation 3.4.1. Let A be a unital ring and let $\phi : A \rightarrow A$ be an automorphism. Define $NK(A, \phi)_+ = \text{hocofiber}(K(A) \rightarrow K(A[t, \phi]))$ and $NK(A, \phi)_- = \text{hocofiber}(K(A) \rightarrow K(A[t, \phi^{-1}]))$. We have

$$K(A[t, \phi]) = K(A) \oplus NK(A, \phi)_+, \quad K(A[t, \phi^{-1}]) = K(A) \oplus NK(A, \phi)_-.$$

Now let A be an arbitrary ring and let $\phi : A \rightarrow A$ be an endomorphism. Write $B = \phi^{-1}A$ for the colimit of the inductive system

$$A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \dots$$

Then ϕ induces an automorphism $\hat{\phi} : B \rightarrow B$ and we can extend it to the unitization \tilde{B} . Put

$$NK(A, \phi)_+ := NK(\tilde{B}, \hat{\phi})_+, \quad NK(A, \phi)_- := NK(\tilde{B}, \hat{\phi})_-,$$

so that $K(\tilde{B}[t, \hat{\phi}]) = K(\tilde{B}) \oplus NK(A, \phi)_+$ and similarly for $K(\tilde{B}[t, \hat{\phi}^{-1}])$. Observe that this definition of $NK(A, \phi)_\pm$ agrees with the above when A is unital and ϕ is an automorphism. Moreover we have $NK(A, \phi)_\pm = NK(B, \hat{\phi})_\pm$.

Lemma 3.5. *Let A be H' -unital, $\phi : A \rightarrow A$ an endomorphism, and $B = \phi^{-1}A$. Then $K(B[t, \phi^{\pm 1}]) \cong \phi^{-1}K(A) \oplus NK(A, \phi)_\pm$.*

Proof. We have an exact sequence

$$0 \rightarrow B[t, \hat{\phi}^{\pm 1}] \rightarrow \widetilde{B[t, \hat{\phi}^{\pm 1}]} \rightarrow \mathbb{Z}[t] \rightarrow 0$$

By Proposition 2.8, the ring $B[t, \hat{\phi}^{\pm 1}]$ is H' -unital. Hence $K(B[t, \hat{\phi}^{\pm 1}]) = K(B) \oplus NK(B, \phi)_\pm$, by excision. Next, the fact that K -theory preserves filtering colimits (see [31, IV.6] for the unital case; the non-unital case follows from the unital case by using that unitization preserves colimits –because it has a right adjoint– and that $K(A) = K(\tilde{A} : A)$) implies that $K(B) \cong \phi^{-1}K(A)$. \square

We shall make use of the construction of the corner skew Laurent polynomial ring $S[t_+, t_-, \phi]$, for a corner-isomorphism $\phi : S \rightarrow pSp$; see [3].

Theorem 3.6. *Let R be a unital ring and let A be a ring. Let $\phi : R \rightarrow pRp$ be a corner-isomorphism. Assume that $R \otimes A$ is H' -unital. Then there is a homotopy fibration of nonconnective spectra*

$$\begin{aligned} K(R \otimes A) &\xrightarrow{1-\phi \otimes 1} K(R \otimes A) \oplus NK(R \otimes A, \phi \otimes 1)_+ \oplus NK(R \otimes A, \phi \otimes 1)_- \\ &\longrightarrow K((R \otimes A)[t_+, t_-, \phi \otimes 1]) \end{aligned}$$

In other words,

$$\begin{aligned} K((R \otimes A)[t_+, t_-, \phi \otimes 1]) &= NK(R \otimes A, \phi \otimes 1)_+ \oplus NK(R \otimes A, \phi \otimes 1)_- \\ &\oplus \text{hocofiber}(K(R \otimes A) \xrightarrow{1-\phi \otimes 1} K(R \otimes A)). \end{aligned}$$

Proof. Step 1: Assume that ϕ is a unital isomorphism and $A = \mathbb{Z}$. In this case the skew Laurent polynomial ring is the crossed product by \mathbb{Z} ; $R[t_+, t_-, \phi] = R[t, t^{-1}, \phi]$. Let $i^\pm : R \rightarrow R[t_\pm, \phi]$ and $j^\pm : R[t_\pm, \phi] \rightarrow R[t_+, t_-, \phi]$ be the inclusion maps. By the proof of [34, Theorem 2.1], there is a homotopy fibration

$$K(R) \oplus K(R) \xrightarrow{\psi} K(R[t_+, \phi]) \oplus K(R[t_-, \phi]) \xrightarrow{[j^+, j^-]} K(R[t_+, t_-, \phi])$$

and $K(R[t_\pm, \phi]) = K(R) \oplus NK(R, \phi)_\pm$. Here

$$\psi = \begin{bmatrix} i^+ & i^+ \\ i^- & i^- \circ \phi \end{bmatrix}$$

Application of Lemma 3.1 yields the fibration of the theorem; this finishes the case when ϕ is a unital isomorphism.

Step 2: Assume that B is an H' -unital ring and that $\phi : B \rightarrow B$ is an isomorphism. Then by the previous step, the augmentation $\tilde{B} \rightarrow \mathbb{Z}$ induces a map of fibration sequences

$$\begin{array}{ccccccc} K(\tilde{B}) & \xrightarrow{1-\tilde{\phi}} & K(\tilde{B}) \oplus NK(\tilde{B}, \tilde{\phi})_+ \oplus NK(\tilde{B}, \tilde{\phi})_- & \longrightarrow & K(\tilde{B}[t_+, t_-, \tilde{\phi}]) \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{Z}) & \xrightarrow{0} & K(\mathbb{Z}) & \xrightarrow{\quad} & K(\mathbb{Z}[t, t^{-1}]) \end{array}$$

Since $B[t_\pm, \phi]$ and $B[t_+, t_-, \phi]$ are H' -unital by Proposition 2.8, the fibers of the vertical maps give the fibration of the theorem.

Step 3: Assume that R is unital and let ϕ be a corner isomorphism. Let A be an H' -unital ring. Write $S = \phi^{-1}R$ for the colimit of the inductive system

$$R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \dots$$

Then ϕ induces an automorphism $\hat{\phi} : S \rightarrow S$. Set $R_n = R$; then $B = S \otimes A = \text{colim}_n R_n \otimes A$ is H' -unital, since $R_n \otimes A$ is H' -unital by hypothesis, and H' -unitality is preserved under filtering colimits (see Remark 2.2). Since $\hat{\phi} \otimes 1$ is an automorphism of B , Step 2 gives

$$(3.7) \quad \begin{aligned} K(B[t_+, t_-, \hat{\phi} \otimes 1]) &= \text{hocofiber}(1 - \hat{\phi} \otimes 1 : K(B) \rightarrow K(B)) \\ &\oplus NK(B, \hat{\phi} \otimes 1)_+ \oplus NK(B, \hat{\phi} \otimes 1)_-. \end{aligned}$$

Because K -theory commutes with filtering colimits, we have $K(B) = (\phi \otimes 1)^{-1}K(R \otimes A)$. Thus by Lemma 3.3,

$$(3.8) \quad \begin{aligned} \text{hocofiber}(1 - \hat{\phi} \otimes 1 : K(B) \rightarrow K(B)) &\cong \\ &\text{hocofiber}(1 - \phi \otimes 1 : K(R \otimes A) \rightarrow K(R \otimes A)) \end{aligned}$$

Write $\varphi_n : R_n \rightarrow S$ for the canonical map of the colimit, and put $e_n = \varphi_n(1)$. For $n \geq 0$, there is a ring isomorphism $\psi_n : (R \otimes A)[t_+, t_-, \phi \otimes 1] \rightarrow (e_n \otimes 1)B[t, t^{-1}, \hat{\phi} \otimes 1](e_n \otimes 1)$, where $e_n \otimes 1 \in S \otimes \tilde{A}$, such that $\psi_n(r \otimes a) = \varphi_n(r) \otimes a$, and $\psi_n(t_+) = (e_n \otimes 1)t(e_n \otimes 1)$, and $\psi_n(t_-) = (e_n \otimes 1)t^{-1}(e_n \otimes 1)$.

Consider the map $\eta : (R \otimes A)[t_+, t_-, \phi \otimes 1] \rightarrow (R \otimes A)[t_+, t_-, \phi \otimes 1]$, $\eta(x) = t_+xt_-$. There is a commutative diagram

$$(3.9) \quad \begin{array}{ccc} (R \otimes A)[t_+, t_-, \phi \otimes 1] & \xrightarrow{\psi_n} & (e_n \otimes 1)B[t, t^{-1}, \hat{\phi} \otimes 1](e_n \otimes 1) \\ \eta \downarrow & & i \downarrow \\ (R \otimes A)[t_+, t_-, \phi \otimes 1] & \xrightarrow{\psi_{n+1}} & (e_{n+1} \otimes 1)B[t, t^{-1}, \hat{\phi} \otimes 1](e_{n+1} \otimes 1) \end{array}$$

It follows that $B[t_+, t_-, \hat{\phi} \otimes 1] = \eta^{-1}(R \otimes A)[t_+, t_-, \phi \otimes 1]$. Hence we have $K(B[t_+, t_-, \hat{\phi} \otimes 1]) \cong \eta^{-1}K((R \otimes A)[t_+, t_-, \phi \otimes 1])$. But since $t_-t_+ = 1$, the map η induces the identity on $K((R \otimes A)[t_+, t_-, \phi \otimes 1])$ (e.g. by [7, 2.2.6]). Thus

$$(3.10) \quad K((R \otimes A)[t_+, t_-, \phi \otimes 1]) \cong K(B[t_+, t_-, \hat{\phi} \otimes 1])$$

In addition we have

$$(3.11) \quad NK(R \otimes A, \phi \otimes 1)_\pm \cong NK(B, \hat{\phi} \otimes 1)_\pm.$$

Rewrite (3.7) using (3.8), (3.11) and (3.10) to finish the third (and final) step. \square

4. MATRICES ASSOCIATED TO FINITE QUIVERS

Let E be a finite quiver. Write E_0 for the set of vertices and E_1 for the set of arrows. In this section we assume both E_0 and E_1 are finite, of cardinalities e_0 and e_1 . If $\alpha \in E_1$, we write $s(\alpha)$ for its source vertex and $r(\alpha)$ for its range. There are two matrices with nonnegative integer coefficients associated with E ; these are best expressed in terms of the *range* and *source* maps $r, s : E_1 \rightarrow E_0$. If $f : E_1 \rightarrow E_0$ is a map of finite sets, and χ_x, χ_y are the characteristic functions of $\{x\}$ and $\{y\}$, we write

$$\begin{aligned} f^* : \mathbb{Z}^{E_0} &\rightarrow \mathbb{Z}^{E_1}, & f^*(\chi_y) &= \sum_{f(x)=y} \chi_x \\ f_* : \mathbb{Z}^{E_1} &\rightarrow \mathbb{Z}^{E_0}, & f_*(\chi_x) &= \chi_{f(x)}. \end{aligned}$$

Put

$$(4.1) \quad M_E = r^*s_* \quad N'_E = s_*r^*$$

We identify these homomorphisms with their matrices with respect to the canonical basis. The matrices $M_E = [m_{\alpha,\beta}] \in M_{e_1}\mathbb{Z}$ and $N'_E = [n_{i,j}] \in M_{e_0}\mathbb{Z}$ are given by

$$m_{\alpha,\beta} = \delta_{r(\alpha),s(\beta)}$$

$$n_{i,j} = \#\{\alpha \in E_1 : s(\alpha) = i, r(\alpha) = j\}$$

For $i = 0, 1$, we consider the chain complex C^i concentrated in degrees 0 and 1, with $C_j^i = \mathbb{Z}^{e_i}$ if $j = 0, 1$, and with boundary map $1 - N'_E$ if $i = 0$ and $1 - M_E$ if $i = 1$. Pictorially

$$(4.2) \quad \begin{aligned} C^0 : \mathbb{Z}^{E_0} &\xrightarrow{1 - N'_E} \mathbb{Z}^{E_0} \\ C^1 : \mathbb{Z}^{E_1} &\xrightarrow{1 - M_E} \mathbb{Z}^{E_1} \end{aligned}$$

Lemma 4.3. *The maps r^* and s_* induce inverse homotopy equivalences $C^0 \leftrightarrows C^1$.*

Proof. Straightforward. \square

Proposition 4.4. *If X is a spectrum, then $\text{hocofiber}(1 - M_E : X^{e_1} \rightarrow X^{e_1}) \cong \text{hocofiber}(1 - N'_E : X^{e_0} \rightarrow X^{e_0})$.*

Proof. Note r^* induces a map

$$\begin{array}{ccccc} X^{e_0} & \xrightarrow{1 - N'_E} & X^{e_0} & \longrightarrow & \text{hocofiber}(1 - N'_E) \\ r^* \downarrow & & r^* \downarrow & & \downarrow f \\ X^{e_1} & \xrightarrow{1 - M_E} & X^{e_1} & \longrightarrow & \text{hocofiber}(1 - M_E) \end{array}$$

From the long exact sequences of homotopy groups of the fibrations above, we obtain

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ H_0(C^0 \otimes \pi_n(X)) & \xrightarrow{r^*} & H_0(C^1 \otimes \pi_n(X)) & & \\ \downarrow & & \downarrow & & \\ \pi_n \text{hocofiber}(1 - N'_E) & \xrightarrow{f} & \pi_n \text{hocofiber}(1 - M_E) & & \\ \downarrow & & \downarrow & & \\ H_1(C^0 \otimes \pi_{n-1}X) & \xrightarrow{r^*} & H_1(C^1 \otimes \pi_{n-1}X) & & \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

By Lemma 4.3, the horizontal maps at the two extremes are isomorphisms; it follows that the map in the middle is an isomorphism too. \square

Recall that a vertex $i \in E_0$ is called a *source* (respectively, a *sink*) in case $r^{-1}(i) = \emptyset$ (respectively, $s^{-1}(i) = \emptyset$). We will denote by $\text{Sink}(E)$ the sets of sinks of E .

5. K-THEORY OF THE LEAVITT ALGEBRA I: FINITE QUIVERS WITHOUT SINKS

Let E be a finite quiver and $M = M_E$. The *path ring* of E is the ring $P = P_{\mathbb{Z}}(E)$ with one generator for each arrow $\alpha \in E_1$ and one generator p_i for each vertex $i \in E_0$, subject to the following relations

$$(5.1) \quad p_i p_j = \delta_{i,j} p_i, \quad (i, j \in E_0)$$

$$(5.2) \quad p_{s(\alpha)} \alpha = \alpha = \alpha p_{r(\alpha)}, \quad (\alpha \in E_1)$$

The ring P has a basis formed by the p_i , the α , and the products $\alpha_1 \cdots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$. We think of these as paths in the quiver, of lengths, 0, 1 and n , respectively. Observe that P is unital, with $1 = \sum_{i \in E_0} p_i$.

Consider the *opposite quiver* E^* ; this is the quiver with the same sets of vertices and arrows, but with the range and source functions switched. Thus $E_i^* = E_i$ ($i = 0, 1$) and if we write α^* for the arrow $\alpha \in E_1$ considered as an arrow of E^* , we have $r(\alpha^*) = s(\alpha)$ and $s(\alpha^*) = r(\alpha)$. The path ring $P^* = P(E^*)$ is generated by the p_i ($i \in E_0$) and the $\alpha^* \in E_1^*$; the relation (5.1) is satisfied, and we also have

$$(5.3) \quad p_{r(\alpha)} \alpha^* = \alpha^* = \alpha^* p_{s(\alpha)}, \quad (\alpha \in E_1).$$

The *Leavitt path ring of E* is the ring $L = L_{\mathbb{Z}}(E)$ on generators p_i ($i \in E_0$), $\alpha \in E_1$, and $\alpha^* \in E_1^*$, subject to relations (5.1), (5.2), and (5.3), and to the following two additional relations

$$(5.4) \quad \alpha^* \beta = \delta_{\alpha, \beta} p_{r(\alpha)}$$

$$(5.5) \quad p_i = \sum_{s(\alpha)=i} \alpha \alpha^* \quad (i \in E_0 \setminus \text{Sink}(E))$$

From these last two relations we obtain

$$(5.6) \quad \begin{aligned} \alpha^* \alpha &= \sum_{s(\beta)=r(\alpha)} \beta \beta^* \\ &= \sum_{\beta \in E_1} m_{\beta, \alpha} \beta \beta^* \end{aligned}$$

It also follows, in case E has no sinks, that the $q_\beta = \beta \beta^*$ are a complete system of orthogonal idempotents; we have

$$(5.7) \quad \sum_{\beta \in E_1} q_\beta = 1, \quad q_\alpha q_\beta = \delta_{\alpha, \beta} q_\beta$$

The ring L is equipped with an involution and a \mathbb{Z} -grading. The involution $x \mapsto x^*$ sends $\alpha \mapsto \alpha^*$ and $\alpha^* \mapsto \alpha$. The grading is determined by $|\alpha| = 1$, $|\alpha^*| = -1$. By [4, proof of Theorem 5.3], we have $L_0 = \bigcup_{n=0}^{\infty} L_{0,n}$, where $L_{0,n}$ is the linear span of all the elements of the form $\gamma \nu^*$, where γ and ν are paths with $r(\gamma) = r(\nu)$ and $|\gamma| = |\nu| = n$. For each i in E^0 , and each $n \in \mathbb{Z}^+$, let us denote by $P(n, i)$ the set of paths γ in E such that $|\gamma| = n$ and $r(\gamma) = i$. The ring $L_{0,0}$ is isomorphic to $\prod_{i \in E^0} k$. In general the ring $L_{0,n}$ is isomorphic to

$$\left[\prod_{m=0}^{n-1} \left(\prod_{i \in \text{Sink}(E)} M_{|P(m, i)|}(\mathbb{Z}) \right) \right] \times \left[\prod_{i \in E_0} M_{|P(n, i)|}(\mathbb{Z}) \right].$$

The transition homomorphism $L_{0,n} \rightarrow L_{0,n+1}$ is the identity on the factors $\prod_{i \in \text{Sink}(E)} M_{|P(m, i)|}(\mathbb{Z})$, for $0 \leq m \leq n-1$, and also on the factor $\prod_{i \in \text{Sink}(E)} M_{|P(n, i)|}(\mathbb{Z})$ of the last term of the displayed formula. The transition homomorphism

$$\prod_{i \in E_0 \setminus \text{Sink}(E)} M_{|P(n, i)|}(\mathbb{Z}) \rightarrow \prod_{i \in E_0} M_{|P(n+1, i)|}(\mathbb{Z})$$

is a block diagonal map induced by the following identification in $L(E)_0$: A matrix unit in a factor $M_{|P(n, i)|}(\mathbb{Z})$, where $i \in E_0 \setminus \text{Sink}(E)$, is a monomial of the form $\gamma \nu^*$, where γ and ν are paths of length n with $r(\gamma) = r(\nu) = i$. Since i is not a sink, we can enlarge the paths γ and ν using the edges that i emits, obtaining paths of length $n+1$, and relation (5.5) in the definition of $L(E)$ gives

$$\gamma \nu^* = \sum_{\{\alpha \in E_1 \mid s(\alpha)=i\}} (\gamma \alpha)(\nu \alpha)^*.$$

Assume E has no sources. For each $i \in E_0$, choose an arrow α_i such that $r(\alpha_i) = i$. Consider the elements

$$t_+ = \sum_{i \in E_0} \alpha_i, \quad t_- = t_+^*$$

One checks that $t_- t_+ = 1$. Thus, since $|t_{\pm}| = \pm 1$, the endomorphism

$$\phi : L \rightarrow L, \quad \phi(x) = t_+ x t_-$$

is homogeneous of degree 0 with respect to the \mathbb{Z} -grading. In particular it restricts to an endomorphism of L_0 . By [3, Lemma 2.4], we have

$$(5.8) \quad L = L_0[t_+, t_-, \phi].$$

For a unital ring A , we may define the Leavitt path A -algebra $L_A(E)$ in the same way as before, with the proviso that elements of A commute with the generators p_i, α, α^* . Observe that

$$(5.9) \quad L_A(E) = L_{\mathbb{Z}}(E) \otimes A.$$

If A is a not necessarily unital ring, we take (5.9) as the definition of $L_A(E)$. We may think of $L_{\mathbb{Z}}(E)$ as the most basic Leavitt path ring.

Let $e'_0 = |\text{Sink}(E)|$. We assume that E_0 is ordered so that the first e'_0 elements of E_0 correspond to its sinks. Accordingly, the first e'_0 rows of the matrix N'_E are 0. Let N_E be the matrix obtained by deleting these e'_0 rows. The matrix that enters the computation of the K -theory of the Leavitt path algebra is

$$\begin{pmatrix} 0 \\ 1_{e_0 - e'_0} \end{pmatrix} - N_E^t: \mathbb{Z}^{e_0 - e'_0} \longrightarrow \mathbb{Z}^{e_0}.$$

By a slight abuse of notation, we will write $1 - N_E^t$ for this matrix. Note that $1 - N_E^t \in M_{e_0 \times (e_0 - e'_0)}(\mathbb{Z})$. Of course $N_E = N'_E$ in case E has no sinks, where N'_E is introduced in Section 4.

Theorem 5.10. *Let A be an H' -unital ring, E a finite quiver, $M = M_E$ and $N = N_E$. Assume the quiver E has no sources. We have*

$$\begin{aligned} K(L_A(E)) \cong & NK(L_0 \otimes A, \phi \otimes 1)_+ \oplus NK(L_0 \otimes A, \phi \otimes 1)_- \\ & \oplus \text{hocofiber}(K(A)^{e_0 - e'_0} \xrightarrow{1 - N^t} K(A)^{e_0}). \end{aligned}$$

Moreover, if in addition E has no sinks then

$$\begin{aligned} K(L_A(E)) \cong & NK(L_0 \otimes A, \phi \otimes 1)_+ \oplus NK(L_0 \otimes A, \phi \otimes 1)_- \\ & \oplus \text{hocofiber}(K(A)^{e_1} \xrightarrow{1 - M^t} K(A)^{e_1}). \end{aligned}$$

Proof. If E has no sinks, then Proposition 4.4 applied to E^* gives

$$\text{hocofiber}(K(A)^{e_1} \xrightarrow{1 - M^t} K(A)^{e_1}) \cong \text{hocofiber}(K(A)^{e_0} \xrightarrow{1 - N^t} K(A)^{e_0})$$

Thus it suffices to prove the first equivalence of the theorem. By (5.8),

$$L_A(E) = (L_0 \otimes A)[t_+, t_-, 1 \otimes \phi]$$

Note $L_0 \otimes A$ is a filtering colimit of rings of matrices with coefficients in A . Since A is H' -unital by hypothesis, each such matrix ring is H' -unital, whence $L_0 \otimes A$ is H' -unital. Hence, by Theorem 3.6

$$\begin{aligned} K(L_A(E)) \cong & NK(L_0 \otimes A, \phi \otimes 1)_+ \oplus NK(L_0 \otimes A, \phi \otimes 1)_- \\ & \oplus \text{hocofiber}(K(L_0 \otimes A) \xrightarrow{1 - \phi \otimes 1} K(L_0 \otimes A)) \end{aligned}$$

As explained in the paragraph immediately above the theorem, we have $L_0 = \bigcup_{n=0}^{\infty} L_{0,n}$. Since E has no sources, it follows that $L_{0,n}$ is the product of exactly $ne'_0 + e_0 = (n+1)e'_0 + (e_0 - e'_0)$ matrix algebras; thus $K(A \otimes L_{0,n}) \cong K(A)^{(n+1)e'_0 + (e_0 - e'_0)}$, since A is H' -unital and K -theory is matrix stable on H' -unital rings (by Theorem 2.6). Moreover the inclusion $L_{0,n} \subset L_{0,n+1}$ induces

$$\Delta_n := \begin{pmatrix} 1_{(n+1)e'_0} & 0 \\ 0 & N^t \end{pmatrix}: K(A)^{(n+1)e'_0 + (e_0 - e'_0)} \longrightarrow K(A)^{(n+1)e'_0 + e_0}.$$

Now, for a path γ on E , we have

$$\phi(\gamma\gamma^*) = \sum_{i,j} \alpha_i \gamma \gamma^* \alpha_j^* = (\alpha_{s(\gamma)} \gamma)(\alpha_{s(\gamma)} \gamma)^*,$$

so that $\phi \otimes 1$ induces

$$\Omega_n := \begin{pmatrix} 0 \\ 1_{ne'_0 + e_0} \end{pmatrix}: K(A)^{ne'_0 + e_0} = K(L_{0,n} \otimes A) \longrightarrow K(A)^{(n+1)e'_0 + e_0}.$$

Summing up, we get a commutative diagram

$$(5.11) \quad \begin{array}{ccccccc} K(L_{0,n} \otimes A) & \xrightarrow{\Delta_n} & K(L_{0,n+1} \otimes A) & \longrightarrow & \cdots & \longrightarrow & K(L_0 \otimes A) \\ \Delta_n - \Omega_n \downarrow & & \downarrow \Delta_{n+1} - \Omega_{n+1} & & & & \downarrow 1 - \phi \otimes 1 \\ K(L_{0,n+1} \otimes A) & \xrightarrow{\Delta_{n+1}} & K(L_{0,n+2} \otimes A) & \longrightarrow & \cdots & \longrightarrow & K(L_0 \otimes A) \end{array}$$

Note that elementary row operations take $\Delta_n - \Omega_n$ to $1_{(n+1)e'_0} \oplus (N^t - 1)$; hence there is an elementary matrix h such that $h(\Delta_n - \Omega_n) = 1_{(n+1)e'_0} \oplus (N^t - 1)$. Moreover one checks that h restricts to the identity on $0 \oplus K(A)^{e_0} \subset K(A)^{(n+1)e'_0 + e_0}$. It follows that the inclusion $i_{n+1} : K(A)^{e_0} \rightarrow 0 \oplus K(A)^{e_0} \subset K(A)^{(n+1)e'_0 + e_0}$ induces an equivalence

$$\begin{aligned} C := \text{hocofiber}(K(A)^{e_0 - e'_0} \xrightarrow{1 - N^t} K(A)^{e_0}) \\ \cong \text{hocofiber}(K(L_{0,n} \otimes A) \xrightarrow{\Delta_n - \Omega_n} K(L_{0,n} \otimes A)), \end{aligned}$$

and that furthermore, the diagram

$$\begin{array}{ccccc} K(L_{0,n} \otimes A) & \xrightarrow{\Delta_n - \Omega_n} & K(L_{0,n+1} \otimes A) & \longrightarrow & C \\ \Omega_n \downarrow & & \Omega_{n+1} \downarrow & & \parallel \\ K(L_{0,n+1} \otimes A) & \xrightarrow{\Delta_{n+1} - \Omega_{n+1}} & K(L_{0,n+2} \otimes A) & \longrightarrow & C \end{array}$$

is homotopy commutative. Hence

$$\begin{array}{ccccc} K(L_{0,n} \otimes A) & \xrightarrow{\Delta_n - \Omega_n} & K(L_{0,n+1} \otimes A) & \longrightarrow & C \\ \Delta_n \downarrow & & \Delta_{n+1} \downarrow & & \parallel \\ K(L_{0,n+1} \otimes A) & \xrightarrow{\Delta_{n+1} - \Omega_{n+1}} & K(L_{0,n+2} \otimes A) & \longrightarrow & C \end{array}$$

is homotopy commutative too. Thus $\text{hocofiber}(1 - 1 \otimes \phi : K(L_0 \otimes A) \rightarrow K(L_0 \otimes A)) \cong C$. \square

6. K -THEORY OF LEAVITT ALGEBRAS II: ROW-FINITE QUIVERS

A quiver E is said to be *row-finite* if for each $i \in E_0$, the set $s^{-1}(i) = \{\alpha \in E_1 \mid s(\alpha) = i\}$ is finite. This is equivalent to saying that the adjacency matrix N'_E of E is a row-finite matrix. For a row-finite quiver E , the Leavitt path algebras $L_{\mathbb{Z}}(E)$ and $L_A(E)$ are defined exactly as in the case of a finite quiver.

Recall that a *complete subgraph* of a quiver E is a subquiver F such that for every $v \in F_0$ either $s_F^{-1}(v) = \emptyset$ or $s_F^{-1}(v) = s_E^{-1}(v)$. If F is a complete subgraph of E , then there is a natural homomorphism $L_A(F) \rightarrow L_A(E)$ (see [4, Lemma 3.2]).

Lemma 6.1. *Let E be a finite quiver and let F be a subquiver of E with $d = |F|$ and $d' = |\text{Sink}(F)|$. Let A be a unital ring. Suppose there is a vertex $v \in E_0 \setminus F_0$ such that $s_E^{-1}(v) \neq \emptyset$ and $r_E(s_E^{-1}(v)) \subseteq F_0$. Consider the subquiver F' of E with $F'_0 = F_0 \cup \{v\}$, $F'_1 = F_1 \cup s_E^{-1}(v)$. Then the following properties hold:*

- (1) $L_A(F)$ is a full corner in $L_A(F')$. In particular $L_A(F)$ and $L_A(F')$ are Morita equivalent.
- (2) $\text{hocofiber}(1 - N_F^t : K(A)^{d-d'} \rightarrow K(A)^d) \cong \text{hocofiber}(1 - N_{F'}^t : K(A)^{d+1-d'} \rightarrow K(A)^{d+1})$.

Proof.

- (1) Set $p = \sum_{i \in F_0} p_i \in L_A(F')$. It is easily seen that $L_A(F) \cong p L_A(F') p$. Since p is a full idempotent in $L_A(F')$, this proves (1).
- (2) Recall that we write $1 - N_F^t$ for the $d \times (d - d')$ -matrix $\begin{pmatrix} 0 \\ 1_{d-d'} \end{pmatrix} - N_F^t$. Note that v is a source in F' , so for every $j \in F'_0$ we have $n_{jv}^{F'} = 0$. The matrices

$$\begin{pmatrix} 0 \\ 1_{d+1-d'} \end{pmatrix} - N_{F'}^t \quad \text{and} \quad \begin{pmatrix} \begin{pmatrix} 0 \\ 1_{d-d'} \end{pmatrix} - N_F^t & 0 \\ 0 & 1 \end{pmatrix}$$

are clearly equivalent by elementary transformations, from which the result follows. \square

For a path $\gamma \in E_n$, with $n \geq 1$, we denote by $v(\gamma)$ the set of all vertices appearing as range or source vertices of the arrows of γ . If $i \in E_0$ is a trivial path, we set $v(i) = \{i\}$. Write $L_E = \{\gamma \in E_* \mid |v(\gamma)| = |\gamma| + 1\}$, the set of paths without repetitions of vertices. Denote by r_{E_*} and s_{E_*} the extensions of r_E and s_E respectively to the set of all paths in E .

Given a quiver with oriented cycles, we define a subquiver \tilde{E} of E by setting $\tilde{E}_0 = \{i \in E_0 \mid r_{E_*}(i) \not\subseteq L_E\}$ and $\tilde{E}_1 = \{\alpha \in E_1 \mid s_E(\alpha) \in \tilde{E}_0\}$. Observe that this is a well-defined quiver because, if $s_E(\alpha) \in \tilde{E}_0$, then $r_E(\alpha) \in \tilde{E}_0$ as well. If E does not have oriented cycles, then we define \tilde{E} as the empty quiver.

Lemma 6.2. *Let E be a quiver. Then \tilde{E} is a complete subgraph of E without sources, and if $\gamma \in E_*$ is a non-trivial closed path then $\gamma \in \tilde{E}_*$.*

Proof. The result is clear in case E does not have oriented cycles. Suppose that E has oriented cycles. By definition, \tilde{E} is a complete subgraph of E . Observe that if $i \in \tilde{E}_0$ then $s_{E_*}^{-1}(i) \subseteq \tilde{E}_*$. Now if $\gamma \in E_*$ is a non-trivial closed path we have $s(\gamma) = r(\gamma) \in \tilde{E}_0$ and so $\gamma \in \tilde{E}_*$.

Pick $v \in \tilde{E}_0$. By construction there is $\gamma = \alpha_1 \cdots \alpha_m \in r_{E_*}(v)$ such that $|v(\gamma)| \leq m$. Hence there exists an index i such that there is a non-trivial closed path based on $r_E(\alpha_i)$. Then $r_E(\alpha_i) \in \tilde{E}_0$ and so $v \in \tilde{E}_0$. Therefore \tilde{E} has no sources. \square

We are now ready to obtain our main general result for a row-finite quiver.

Theorem 6.3. *Let A be either a ring with local units or an H' -unital ring which is torsion free as a \mathbb{Z} -module, and let E be a row-finite quiver. Then there is a map*

$$\text{hocofiber}(K(A)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} K(A)^{(E_0)}) \rightarrow K(L_A(E)),$$

which induces a naturally split monomorphism at the level of homotopy groups

$$(6.4) \quad \pi_*(\text{hocofiber}(K(A)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1 - N_E^t} K(A)^{(E_0)})) \rightarrow K_*(L_A(E)).$$

Proof. We first deal with the case of a finite quiver E . Set $d = |E_0|$ and $d' = |\text{Sink}(E)|$.

Consider the subquiver F of E given by $F_0 = \tilde{E}_0 \cup \text{Sink}(E)$ and $F_1 = \tilde{E}_1$. Using Lemma 6.2 we see that F is a complete subgraph of E such that every non-trivial closed path on E has all its arrows and vertices in F . Moreover we have $\text{Sink}(F) = \text{Sink}(E)$.

Set $p = |F_0|$ and $k = d - p$. Suppose that $k > 0$. In this case we will build a chain of complete subgraphs of E , $F = F^0 \subset F^1 \subset \cdots \subset F^k = E$, with $|F_0^{i+1} \setminus F_0^i| = 1$, and such that the following conditions hold for every $i = 0, \dots, k-1$:

- (i) $\text{Sink}(F^i) = \text{Sink}(E)$.
- (ii) $L_{\mathbb{Z}}(F^i)$ is a full corner in $L_{\mathbb{Z}}(F^{i+1})$.
- (iii)

$$\begin{aligned} & \text{hocofiber}(\begin{pmatrix} 0 \\ 1_{p+i-d'} \end{pmatrix} - N_{F^i}^t : K(A)^{p+i-d'} \longrightarrow K(A)^{p+i}) \\ & \cong \text{hocofiber}(\begin{pmatrix} 0 \\ 1_{p+i+1-d'} \end{pmatrix} - N_{F^{i+1}}^t : K(A)^{p+i+1-d'} \longrightarrow K(A)^{p+i+1}). \end{aligned}$$

Suppose we have defined F^i for $0 \leq i < k$. We are going to define F^{i+1} . We first show that there is a vertex $v \in E_0 \setminus F_0^i$ such that $r_E(s_E^{-1}(v)) \subseteq F_0^i$. Pick $v_1 \in E_0 \setminus F_0^i$. Since $\text{Sink}(F^i) = \text{Sink}(E)$ we have that $s_E^{-1}(v_1) \neq \emptyset$. If there exists $\alpha_1 \in s_E^{-1}(v_1)$ such that $r_E(\alpha_1) \notin F_0^i$, set $v_2 = r_E(\alpha_1)$. Since the number of vertices in $E_0 \setminus F_0^i$ is finite, proceeding in this way we will get either a vertex $v \in E_0 \setminus F_0^i$ such that $r_E(s_E^{-1}(v)) \subseteq F_0^i$ or a path $\gamma = \alpha_1 \alpha_2 \cdots \alpha_m$ with $\alpha_j \in E_1 \setminus F_1^i$ such that $r_E(\alpha_m) \in \{r_E(\alpha_1), \dots, r_E(\alpha_{m-1})\}$. But the latter case cannot occur: the path γ would not belong to L_E and consequently we would obtain $r_E(\alpha_m) \in \tilde{E}_0 \subseteq F_0^i$, a contradiction. Therefore we put $F_0^{i+1} = F_0^i \cup \{v\}$ and $F_1^{i+1} = F_1^i \cup s_E^{-1}(v)$. By construction we get (i) and that F^{i+1} is a complete subgraph of E , and (ii) and (iii) follow from Lemma 6.1.

Set $\ell = |\{v \in \text{Sink}(E) \mid r_{E_*}^{-1}(v) \subseteq L_E\}|$. Then we clearly have $K(L_A(F)) \cong K(L_A(\tilde{E})) \oplus K(A)^\ell$. Now by Lemma 6.2 \tilde{E} is a quiver without sources. Note that $|\tilde{E}_0| - |\text{Sink}(\tilde{E})| = (p - \ell) - (d' - \ell) = p - d'$, so from

Theorem 5.10 we get a decomposition

$$\begin{aligned} K(L_A(\tilde{E})) = & NK(L_0(\tilde{E}) \otimes A, \phi \otimes 1)_+ \oplus NK(L_0(\tilde{E}) \otimes A, \phi \otimes 1)_- \oplus \\ & \text{hocofiber}(\begin{pmatrix} 0 \\ 1_{p-d'} \end{pmatrix} - N_{\tilde{E}}^t : K(A)^{p-d'} \rightarrow K(A)^{p-\ell}). \end{aligned}$$

Hence

$$\begin{aligned} (6.5) \quad K(L_A(F)) \cong & K(L_A(\tilde{E})) \oplus K(A)^\ell \\ \cong & NK(L_0(\tilde{E}) \otimes A, \phi \otimes 1)_+ \oplus NK(L_0(\tilde{E}) \otimes A, \phi \otimes 1)_- \\ & \oplus \text{hocofiber}(\begin{pmatrix} 0 \\ 1_{p-d'} \end{pmatrix} - N_F^t : K(A)^{p-d'} \rightarrow K(A)^p). \end{aligned}$$

This gives the result for $F^0 = F$. Applying inductively (ii) and (iii) to the quivers of the chain $F = F^0 \subset F^1 \subset \dots \subset F^k = E$, and using Lemma 2.7, we get the assertions of theorem for finite E . Let E be a row-finite quiver. By [4, Lemma 3.2], E is the filtered colimit of its finite complete subgraphs. Since filtered colimits are exact, hocofiber commutes with them, so we get the monomorphism in (6.4). To compute the cokernel of this map, note that the construction of the graph \tilde{E} is functorial in the category of row-finite quivers and complete graph homomorphisms. Moreover we get $\tilde{E} = \text{colim } \tilde{F}$, where F ranges on the family of all finite complete subquivers of E . For each $i \in \tilde{E}_0$ we select an arrow $\alpha_i \in \tilde{E}_1$ such that $r(\alpha_i) = i$. This choice induces a compatible choice of arrows in the quivers \tilde{F} corresponding to finite complete subquivers F of E . Hence, if $F^1 \subseteq F^2$ are two finite complete subquivers of E , then the corresponding corner-isomorphisms ϕ^i on $L(\tilde{F}^i)_0$ satisfy that $\phi^2|_{L(\tilde{F}^1)_0} = \phi^1$, and thus we obtain maps

$$\kappa_\pm : NK(L(F^1)_0 \otimes A, \phi_1 \otimes 1)_\pm \longrightarrow NK(L(F^2)_0 \otimes A, \phi_2 \otimes 1)_\pm$$

such that the map $K(L_A(F^1)) \rightarrow K(L_A(F^2))$, written in terms of the decomposition given in Theorem 5.10, is of the form $\kappa_+ \oplus \kappa_- \oplus \kappa$, where κ is the map between the corresponding hocofiber terms. The result follows. \square

Remark 6.6. The proof above shows that cokernel of the map (6.4) can be expressed in terms of twisted nil- K -groups. If E is finite, the cokernel is $NK_*(L_0(\tilde{E}) \otimes A, \phi \otimes 1)_+ \oplus NK_*(L_0(\tilde{E}) \otimes A, \phi \otimes 1)_-$, by (6.5). In the general case, it is the colimit of the cokernels corresponding to each of the finite complete subquivers.

7. LEAVITT RINGS WITH REGULAR SUPERCOHERENT COEFFICIENTS

In this section we will determine the K -theory of the Leavitt path ring of a row-finite quiver over a regular supercoherent ring k .

Recall that a unital ring R is said to be *coherent* if its finitely presented modules form an abelian subcategory of the category of all modules. We say that R is *regular coherent* if it is coherent and in addition any finitely presented module has finite projective dimension. Equivalently R is regular coherent if any finitely presented module has a finite resolution by finitely generated projective modules. The ring R is called *supercoherent* in case all polynomial rings $R[t_1, \dots, t_p]$ are coherent, see [17]. Note that every Noetherian ring is supercoherent. A more general version of regularity was introduced by Vogel, see [5]. We will call this concept *Vogel-regularity*. For a coherent ring R , Vogel-regularity agrees with regularity ([5, Proposition 10]). Since Vogel-regularity is stable under the formation of polynomial rings ([5, Proposition 5(3)]), it follows that $R[t_1, \dots, t_p]$ is regular for every p in case R is regular supercoherent. Observe also that any flat universal localization $R \rightarrow R\Sigma^{-1}$ of a regular (super)coherent ring is also regular (super)coherent. This is due to the fact that every finitely presented $R\Sigma^{-1}$ -module is induced from a finitely presented R -module ([24, Corollary 4.5]). In particular all the rings $R[t_1, t_1^{-1}, \dots, t_p, t_p^{-1}]$ are regular supercoherent if R is regular supercoherent.

Next we will compute the K -theory of the Leavitt algebra of a quiver E over a supercoherent coefficient ring k . As a first step, we consider the case where E is finite and without sources.

Proposition 7.1. *Let E be a finite quiver without sources and let k be a regular supercoherent ring. Let $B = \phi^{-1}L_0$, where L_0 is the homogeneous component of degree 0 of $L_k(E)$. Let $D = B \oplus k$ be the k -unitization of B . Then D is regular supercoherent.*

Proof. Since the ring corresponding to $k[t_1, \dots, t_p]$ is $D[t_1, \dots, t_p]$, it suffices to show that D is regular coherent whenever k is so.

We are going to apply [17, Proposition 1.6]: If $R = \text{colim}_{i \in I} R_i$, where I is a filtering poset, the ring R is a flat left R_i -module for all $i \in I$, and each R_i is regular coherent, then R is regular coherent.

We will show that L_0 is flat as a left $L_{0,n}$ -module. It is enough to show that $L_{0,n+1}$ is flat over $L_{0,n}$. Observe that

$$L_{0,n+1} = \bigoplus_{|\gamma| \leq n, r(\gamma) \in \text{Sink}(E)} L_{0,n} \gamma \gamma^* \bigoplus_{|\gamma|=n+1} L_{0,n+1} \gamma \gamma^*,$$

so that we only need to analyse the terms $L_{0,n+1} \gamma \gamma^*$ with $\gamma \in E_{n+1}$. Write $\gamma = \gamma_0 \alpha$ with $\gamma_0 \in E_n$ and $\alpha \in E_1$. For $v \in E_0$ set

$$Z_{v,n} = \{\beta \in E_1 \mid r(\beta) = v \text{ and there exists } \eta \in E_n \text{ such that } r(\eta) = s(\beta)\}.$$

For each $\beta \in Z_{v,n}$, select $\eta_\beta \in E_n$ such that $r(\eta_\beta) = s(\beta)$. Then

$$L_{0,n+1} \gamma \gamma^* = \bigoplus_{\beta \in Z_{r(\alpha),n}} L_{0,n} \eta_\beta \beta \alpha^* (\gamma_0)^* \cong \bigoplus_{\beta \in Z_{r(\alpha),n}} L_{0,n} \eta_\beta (\eta_\beta)^*.$$

Thus $L_{0,n+1}$ is indeed projective as a $L_{0,n}$ -module.

By [17, Proposition 1.6] we get that L_0 is regular coherent. Now observe that $D = \text{colim}(e_i B e_i \oplus k)$, where e_i is the image of $1 \in L_0$ through the canonical map $\varphi_i: L_0 \rightarrow B$ to the colimit. Since $e_i B e_i \cong L_0$ is unital, we get that $e_i B e_i \oplus k \cong L_0 \times k$, where $L_0 \times k$ denotes the ring direct product of L_0 and k , and so it is regular coherent by the above. By another application of [17, Proposition 1.6], it suffices to check that $e_{i+1} B e_{i+1} \oplus k$ is flat as a left $e_i B e_i \oplus k$ -module, which in turn is equivalent to checking that L_0 is flat as a left $(1-e)k \times e L_0 e$ -module, where $e = \phi(1) = \sum_{i \in E_0} \alpha_i \alpha_i^*$. Recall that, for $i \in E_0$, $\alpha_i \in E_1$ is such that $r(\alpha_i) = i$. We have $L_0 = (1-e)L_0 \oplus eL_0$ and since $(1-e)L_0$ is flat as a left $(1-e)k$ -module, it suffices to show that eL_0 is flat as a left $eL_0 e$ -module. Because

$$L_{0,1} \cong k^{\text{Sink}(E)} \times \prod_{i \in E_0} M_{|P(1,i)|}(\mathbb{Z})$$

we see that there is a central idempotent z in L_0 such that $e \in zL_0$ and e is a full idempotent in zL_0 , that is $zL_0 = zL_0 e L_0$. Now a standard argument shows that eL_0 is indeed projective as a left $eL_0 e$ -module. Indeed there exists $n \geq 1$ and a finitely generated projective L_0 -module P such that

$$zL_0 \oplus P \cong (L_0 e)^n;$$

tensoring this with eL_0 we get $eL_0 \oplus eP \cong (eL_0 e)^n$, as wanted. This concludes the proof. \square

Our next lemma follows essentially from Waldhausen [30].

Lemma 7.2. *Let R be a regular supercoherent ring and let ϕ be an automorphism of R . Extend ϕ to an automorphism of $R[t_1, t_1^{-1}, \dots, t_p, t_p^{-1}]$ by $\phi(t_i) = t_i$. Then $NK_n(R[t_1, t_1^{-1}, \dots, t_p, t_p^{-1}], \phi)_\pm = 0$ for every $p \geq 0$ and every $n \in \mathbb{Z}$.*

Proof. For $n \geq 1$ this follows from [30, Theorem 4], because, as we observed before, $R[t_1, t_1^{-1}, \dots, t_p, t_p^{-1}]$ is regular coherent. Let $n \leq 1$ and assume that $NK_i(R[t_1, t_1^{-1}, \dots, t_p, t_p^{-1}], \phi)_+ = 0$ for every $p \geq 0$, for every $i \geq n$, and for every automorphism ϕ of R . To show the result for NK_{n-1} , it will be enough to show that $NK_{n-1}(R, \phi)_+ = 0$. Since $R[t, t^{-1}]$ is regular supercoherent we have

$$K_n((R[t, t^{-1}])[s, \phi]) = K_n(R[t, t^{-1}]) \oplus NK_n(R[t, t^{-1}], \phi) = K_n(R[t, t^{-1}]),$$

by induction hypothesis. It follows that

$$(7.3) \quad K_n(R[t, t^{-1}][s, \phi]) = K_n(R) \oplus K_{n-1}(R)$$

because $NK_n(R) = 0$ again by induction hypothesis. On the other hand we have

$$(7.4) \quad \begin{aligned} K_n(R[s, \phi][t, t^{-1}]) &= K_n(R[s, \phi]) \oplus K_{n-1}(R[s, \phi]) \oplus NK_n(R[s, \phi])^2 \\ &= K_n(R) \oplus K_{n-1}(R) \oplus NK_{n-1}(R, \phi)_+ \oplus NK_n(R[s, \phi])^2. \end{aligned}$$

Comparison of (7.3) and (7.4) gives

$$NK_{n-1}(R, \phi)_+ = 0 = NK_n(R[s, \phi]),$$

as desired. \square

Proposition 7.5. *Let k be a regular supercoherent ring and let E be a finite quiver without sources. Set $d = |E_0|$ and $d' = |\text{Sink}(E)|$. Then*

$$K(L_k(E)) \cong \text{hocofiber}(K(k)^{d-d'} \xrightarrow{1-N_E^t} K(k)^d).$$

Proof. Let $B = \phi^{-1}L_0$, where $\phi = \phi \otimes 1: L_0 = L_0^{\mathbb{Z}} \otimes k \rightarrow L_0 = L_0^{\mathbb{Z}} \otimes k$ is the corner-isomorphism defined by $\phi(x) = t_+xt_-$, as in Section 5. Note that since k is regular supercoherent and B is H' -unital we have $NK(\tilde{B}, \hat{\phi})_{\pm} = NK(B \oplus k, \hat{\phi})_{\pm}$, where $B \oplus k$ denotes the k -unitization of B . Now it follows from Proposition 7.1 that $B \oplus k$ is regular supercoherent. Therefore Lemma 7.2 gives that $NK(B \oplus k, \hat{\phi})_{\pm} = 0$. It follows that $NK(L_0, \phi)_{\pm} = NK(\tilde{B}, \hat{\phi})_{\pm} = NK(B \oplus k, \hat{\phi})_{\pm} = 0$ and so the result follows from Theorem 5.10. \square

Theorem 7.6. *Let k be a regular supercoherent ring and let E be a row-finite quiver. Then*

$$K(L_k(E)) \cong \text{hocofiber}(K(k)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1-N_E^t} K(k)^{(E_0)}).$$

It follows that there is a long exact sequence

$$\begin{aligned} K_n(k)^{(E_0 \setminus \text{Sink}(E))} &\xrightarrow{1-N_E^t} K_n(k)^{(E_0)} \\ &\longrightarrow K_n(L_k(E)) \longrightarrow K_{n-1}(k)^{(E_0 \setminus \text{Sink}(E))}. \end{aligned}$$

Proof. The case when E is finite follows from Proposition 7.5 and the argument of the proof of Theorem 6.3. The general case follows from the finite case, by the same argument as that given for the proof of 6.3. \square

Corollary 7.7. *Let k be a principal ideal domain and let E be a row-finite quiver. Then*

$$K_0(L_k(E)) \cong \text{coker}(1 - N^t: \mathbb{Z}^{(E_0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E_0)}),$$

and

$$\begin{aligned} K_1(L_k(E)) &\cong \text{coker}(1 - N^t: K_1(k)^{(E_0 \setminus \text{Sink}(E))} \longrightarrow K_1(k)^{(E_0)}) \\ &\oplus \ker(1 - N^t: \mathbb{Z}^{(E_0 \setminus \text{Sink}(E))} \longrightarrow \mathbb{Z}^{(E_0)}). \end{aligned}$$

Remark 7.8. If we only assume that k is regular coherent in Theorem 7.6, then the long exact sequence in the statement terminates at $K_0(L_k(E))$, although conjecturally the long exact sequence should still stand under this weaker hypothesis on k , see [5].

8. HOMOTOPY ALGEBRAIC K -THEORY OF THE LEAVITT ALGEBRA

Homotopy algebraic K -theory, introduced by C. Weibel in [32], is a particularly well-behaved variant of algebraic K -theory: it is polynomial homotopy invariant, excisive, Morita invariant, and preserves filtering colimits. There is a comparison map

$$(8.1) \quad K_*(A) \rightarrow KH_*(A).$$

It is proved in [32] that if A is unital and $K_n(A) \rightarrow K_n(A[t_1, \dots, t_p])$ is an isomorphism for all $p \geq 1$ (i.e. A is K_n -regular) then (8.1) is an isomorphism for $* \leq n$. In particular if A is unital and K -regular, that is, if it is K_n -regular for all n , then (8.1) is an isomorphism for all $* \in \mathbb{Z}$. Further, we have:

Lemma 8.2. *Let A be a H' -unital ring, torsion free as a \mathbb{Z} -module. If A is K_n -regular, then $K_m(A) \rightarrow KH_m(A)$ is an isomorphism for all $m \leq n$.*

Proof. By Remark 2.2, $A[t_1, \dots, t_p]$ is H' -unital for all p . Hence the split exact sequence of rings

$$0 \rightarrow A[t_1, \dots, t_p] \rightarrow \tilde{A}[t_1, \dots, t_p] \rightarrow \mathbb{Z}[t_1, \dots, t_p] \rightarrow 0$$

induces a decomposition $K_*(\tilde{A}[t_1, \dots, t_p]) = K_*(\mathbb{Z}) \oplus K_*(A[t_1, \dots, t_p])$, since \mathbb{Z} is K -regular. Thus \tilde{A} is K_n -regular, and therefore $K_m(\tilde{A}) = KH_m(\tilde{A}) = KH_m(A) \oplus K_m(\mathbb{Z})$ for $m \leq n$. Splitting off the summand $K_m(\mathbb{Z})$, we get the result. \square

Example 8.3. Examples of K -regular rings include regular supercoherent rings (see [30, Theorem 4]), and both stable and commutative C^* -algebras (see [23, 3.4, 3.5] and [16, 5.3]). A theorem of Vorst (see [29]) says that if a unital ring R is K_n -regular, then it is K_m -regular for all $m \leq n$. If R is commutative unital and of finite type over a field of characteristic zero, then R is $K_{-\dim R}$ -regular ([9]).

Theorem 8.4. *Let R be a unital ring and let A be a ring. Let $\phi : R \rightarrow pRp$ be a corner-isomorphism. Then*

$$KH((R \otimes A)[t_+, t_-, \phi \otimes 1]) \cong \text{hocofiber}(KH(R \otimes A) \xrightarrow{1-\phi \otimes 1} KH(R \otimes A)).$$

Proof. We shall assume that $A = \mathbb{Z}$ and ϕ is an isomorphism; the general case follows from this by the same argument as in the proof of Theorem 3.6, keeping in mind that KH satisfies excision for all (not necessarily H' -unital) rings. By [10, Thm. 6.6.2] there exist a triangulated category kk and a functor $j : \text{Rings} \rightarrow kk$ which is matrix invariant and polynomial homotopy invariant, sends short exact sequences of rings to exact triangles, and is universal initial among all such functors. Hence the functor $\text{Rings} \rightarrow \text{Ho}(\text{Spectra})$, $A \mapsto KH(A)$, factors through an exact functor $\overline{KH} : kk \rightarrow \text{Ho}(\text{Spectra})$. By [10, Thm. 7.4.1], there is an exact triangle in kk

$$R \xrightarrow{1-\phi} R \longrightarrow R[t, t^{-1}, \phi] \longrightarrow \Sigma R$$

Applying \overline{KH} we get an exact triangle

$$KH(R) \xrightarrow{1-\phi} KH(R) \longrightarrow KH(R[t, t^{-1}, \phi]) \longrightarrow \Sigma KH(R).$$

□

Lemma 8.5. *Let R be a unital ring, $e \in R$ an idempotent. Assume e is full. Further let A be any ring. Then the inclusion map $eRe \otimes A \rightarrow R \otimes A$ induces an equivalence $KH(eRe \otimes A) \rightarrow KH(R \otimes A)$.*

Proof. By definition, $KH(R) = |[n] \rightarrow K(R[t_0, \dots, t_n]/\langle t_0 + \dots + t_n - 1 \rangle)|$. The case $A = \mathbb{Z}$ follows from 2.7 applied to each of the polynomial rings $R[t_0, \dots, t_n]/\langle t_0 + \dots + t_n - 1 \rangle$. As in the proof of Lemma 2.7, the general case follows from the case $A = \mathbb{Z}$ by excision. □

Theorem 8.6. *Let A be a ring, and E a row-finite quiver. Then*

$$KH(L_A(E)) \cong \text{hocofiber}(KH(A)^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1-N_E^t} KH(A)^{(E_0)}).$$

Proof. The case when E is finite and has no sources follows from Theorem 8.4 using the argument of the proof of Theorem 5.10. The case for arbitrary finite E follows as in the proof of Theorem 6.3, substituting Lemma 8.5 for 2.7. The general case follows from the finite case by the same argument as in 6.3. □

Example 8.7. As an application of the theorem above, consider the case when E is the quiver with one vertex and $n+1$ loops. In this case, $L_{\mathbb{Z}}(E) = L_{1,n}$ is the classical Leavitt ring [21], and $N_E^t = [n+1]$. Hence by Theorem 8.4, we get that $KH(A \otimes L_{1,n})$ is KH with \mathbb{Z}/n -coefficients:

$$(8.8) \quad KH_*(A \otimes L_{1,n}) = KH_*(A, \mathbb{Z}/n)$$

Thus the effect on KH of tensoring with $L_{1,n}$ is similar to the effect on K^{top} of tensoring a C^* -algebra with the Cuntz algebra \mathcal{O}_{n+1} ([11], [12]). If A is a $\mathbb{Z}[1/n]$ -algebra, then $KH_*(A, \mathbb{Z}/n) = K_*(A, \mathbb{Z}/n)$ [32, 1.6], so we may substitute K -theory for homotopy K -theory in the right hand side of (8.8).

9. COMPARISON WITH THE K -THEORY OF CUNTZ-KRIEGER ALGEBRAS

In this section we consider the Cuntz-Krieger C^* -algebra $C^*(E)$ associated to a row-finite quiver E . If \mathfrak{A} is any C^* -algebra, we write $C_{\mathfrak{A}}^*(E) = C^*(E) \bar{\otimes} \mathfrak{A}$ for the C^* -algebra tensor product. Since $C^*(E)$ is nuclear, there is no ambiguity on the C^* -norm we are using here. Define a map $\gamma_n^{\mathfrak{A}} = \gamma_n^{\mathfrak{A}}(E)$ so that the following diagram commutes

$$\begin{array}{ccc} K_n(C_{\mathfrak{A}}^*(E)) & \longrightarrow & KH_n(C_{\mathfrak{A}}^*(E)) \\ \uparrow & & \downarrow \\ K_n(L_{\mathfrak{A}}(E)) & \xrightarrow{\gamma_n^{\mathfrak{A}}} & K_n^{\text{top}}(C_{\mathfrak{A}}^*(E)) \end{array}$$

The purpose of this section is to analyze when the map $\gamma_n^{\mathfrak{A}}$ is an isomorphism.

The following is the spectrum-level version of a result of Cuntz and Krieger [14], [13], later generalized by others; see e.g. [22, Theorem 3.2].

Theorem 9.1. *Let \mathfrak{A} be a C^* -algebra and E a row-finite quiver. Then*

$$K^{\text{top}}(C_{\mathfrak{A}}^*(E)) = \text{hocofiber}(\ K^{\text{top}}(\mathfrak{A})^{(E_0 \setminus \text{Sink } E)} \xrightarrow{1 - N_E^t} K^{\text{top}}(\mathfrak{A})^{(E_0)} \)$$

Proof. The proof follows the same steps as the one of Theorem 8.6. In particular, the same arguments allow us to reduce to the case of a finite quiver E with no sources. In this case essentially the same proof as in [13, Proposition 3.1] applies. Namely, note that $L_{\mathfrak{A}}(E)$ is isomorphic to a dense $*$ -subalgebra of $C_{\mathfrak{A}}^*(E)$, and let \mathcal{F} be the norm completion of $L_0(E) \otimes \mathfrak{A}$ in $C_{\mathfrak{A}}^*(E)$. Then $\mathcal{K} \bar{\otimes} C_{\mathfrak{A}}^*(E)$ is a crossed product of $\mathcal{K} \bar{\otimes} \mathcal{F}$ by an automorphism $\hat{\phi}$, and Pimsner-Voiculescu gives an exact triangle

$$\mathcal{K} \bar{\otimes} \mathcal{F} \xrightarrow{1 - \hat{\phi}} \mathcal{K} \bar{\otimes} \mathcal{F} \longrightarrow \mathcal{K} \bar{\otimes} C_{\mathfrak{A}}^*(E) \longrightarrow \Sigma(\mathcal{K} \bar{\otimes} \mathcal{F})$$

in KK . Now stability gives the following exact triangle in KK :

$$(9.2) \quad \mathcal{F} \xrightarrow{1 - \phi} \mathcal{F} \longrightarrow C_{\mathfrak{A}}^*(E) \longrightarrow \Sigma \mathcal{F}$$

where ϕ is just a corner-isomorphism. Since $C^*\text{-alg} \longrightarrow \text{KK}$ is universal amongst all stable, homotopy invariant, half-exact for cpc-split extensions functors to a triangulated category and

$$C^*\text{-alg} \longrightarrow \text{Ho}(\text{Spectra}), \quad A \mapsto K^{\text{top}}(A)$$

is one such functor which in addition maps mapping cone triangles to exact triangles in $\text{Ho}(\text{Spectra})$, the exact triangle (9.2) is exact in $\text{Ho}(\text{Spectra})$; see [15, Theorem 8.27]. But just as in the proof of Theorem 5.10, we get

$$\begin{aligned} & \text{hocofiber}(\ K^{\text{top}}(\mathcal{F}) \xrightarrow{1 - \phi} K^{\text{top}}(\mathcal{F}) \) \\ & \cong \text{hocofiber}(\ K^{\text{top}}(\mathfrak{A})^{(E_0 \setminus \text{Sink } E)} \xrightarrow{1 - N_E^t} K^{\text{top}}(\mathfrak{A})^{(E_0)} \) \end{aligned}$$

This concludes the proof. \square

Corollary 9.3. *Assume $K_*(\mathfrak{A}) \rightarrow K_*^{\text{top}}(\mathfrak{A})$ is an isomorphism for $* = n, n-1$. Then $\gamma_n^{\mathfrak{A}}$ is a split surjection. If in addition $K_*(\mathfrak{A}) \rightarrow KH_*(\mathfrak{A})$ and $K_*(L_{\mathfrak{A}}(E)) \rightarrow KH_*(L_{\mathfrak{A}}(E))$ are isomorphisms for $* = n, n-1$, then γ_n is an isomorphism.*

Proof. We have

$$\begin{aligned} \pi_n(\text{hocofiber}(\ K(\mathfrak{A})^{(E_0 \setminus \text{Sink } E)} \xrightarrow{1 - N_E^t} K(\mathfrak{A})^{(E_0)} \)) \\ \cong \pi_n(\text{hocofiber}(\ K^{\text{top}}(\mathfrak{A})^{(E_0 \setminus \text{Sink } E)} \xrightarrow{1 - N_E^t} K^{\text{top}}(\mathfrak{A})^{(E_0)} \)) \end{aligned}$$

by the five lemma. Next apply Theorems 6.3 and 9.1 to obtain the first assertion. For the second assertion, use Theorem 8.6. \square

Theorem 9.4. *Let E be a finite quiver without sinks. Assume that $\det(1 - N_E^t) \neq 0$. Then $\gamma_n^{\mathbb{C}}$ is an isomorphism for $n \geq 0$ and the zero map for $n \leq -1$.*

Proof. Because \mathbb{C} is regular supercoherent, we have

$$(9.5) \quad K(L_{\mathbb{C}}(E)) \cong \text{hocofiber}(\ K(\mathbb{C})^{(E_0)} \xrightarrow{1 - N_E^t} K(\mathbb{C})^{(E_0)} \),$$

by Theorem 7.6. Thus $K_n(L_{\mathbb{C}}(E)) = 0$ for $n \leq -1$, and $\gamma_0^{\mathbb{C}}$ is an isomorphism by the five lemma. Moreover if $n = |\det(1 - N_E^t)|$, then $n^2 K_*(L_{\mathbb{C}}(E)) = 0$, by (9.5). Hence the sequence

$$(9.6) \quad 0 \rightarrow K_m(L_{\mathbb{C}}(E)) \rightarrow K_m(L_{\mathbb{C}}(E), \mathbb{Z}/n^2) \rightarrow K_{m-1}(L_{\mathbb{C}}(E)) \rightarrow 0$$

is exact for all m . On the other hand, by (9.5) and Theorem 9.1, we have a map of exact sequences ($m \in \mathbb{Z}$)

$$\begin{array}{ccc}
 K_m(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} & \longrightarrow & K_m^{\text{top}}(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} \\
 \downarrow & & \downarrow \\
 K_m(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} & \longrightarrow & K_m^{\text{top}}(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} \\
 \downarrow & & \downarrow \\
 K_m(L_{\mathbb{C}}(E), \mathbb{Z}/n^2) & \longrightarrow & K_m^{\text{top}}(C_{\mathbb{C}}^*(E), \mathbb{Z}/n^2) \\
 \downarrow & & \downarrow \\
 K_{m-1}(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} & \longrightarrow & K_{m-1}^{\text{top}}(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} \\
 \downarrow & & \downarrow \\
 K_{m-1}(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)} & \longrightarrow & K_{m-1}^{\text{top}}(\mathbb{C}, \mathbb{Z}/n^2)^{(E_0)}
 \end{array}$$

By a theorem of Suslin [27] the comparison map $K_m(\mathbb{C}, \mathbb{Z}/q) \rightarrow K_m^{\text{top}}(\mathbb{C}, \mathbb{Z}/q)$ is an isomorphism for $m \geq 0$ and $q \geq 1$. Hence the map $K_*(L_{\mathbb{C}}(E), \mathbb{Z}/q) \rightarrow K_*^{\text{top}}(L_{\mathbb{C}}(E), \mathbb{Z}/q)$ is an isomorphism, by Theorems 7.6 and 9.1. Combine this together with (9.6) and induction to finish the proof. \square

Remark 9.7. Chris Smith, a student of Gene Abrams, has given a geometric characterization of those finite quivers E with no sinks which satisfy $\det(1 - N_E^t) \neq 0$ [25].

Example 9.8. It follows from the theorem above that the map $\gamma_n^{\mathfrak{A}}$ is an isomorphism for every finite dimensional C^* -algebra \mathfrak{A} . Let $\{\mathfrak{A}_n \rightarrow \mathfrak{A}_{n+1}\}_n$ be an inductive system of finite dimensional C^* -algebras; write A and \mathfrak{A} for its algebraic and its C^* -colimit. Because K -theory commutes with algebraic filtering colimits and K^{top} commutes with C^* -filtering colimits, we conclude that, for E as in the theorem above, the map $K_*(L_A(E)) \rightarrow K_*(L_{\mathfrak{A}}(E))$ is an isomorphism for $* \geq 0$.

Remark 9.9. Let E be a finite quiver with sinks, $\tilde{E} \subset E$ as in Lemma 6.2, and $F = \tilde{E} \cup \text{Sink}(E)$. Then, by Theorem 7.6 and the proof of Theorem 6.3, $K_n(L_{\mathbb{C}}(E)) = K_n(L_{\mathbb{C}}(F)) = K_n(L_{\mathbb{C}}(\tilde{E})) \oplus K_n(\mathbb{C})^{\text{Sink}(E)}$. Similarly,

$$K_n^{\text{top}}(C_{\mathbb{C}}^*(E)) = K_n^{\text{top}}(C_{\mathbb{C}}^*(\tilde{E})) \oplus K_n^{\text{top}}(\mathbb{C})^{\text{Sink}(E)}.$$

By naturality, $\gamma_n^{\mathbb{C}}$ restricts on $K_n(\mathbb{C})^{\text{Sink}(E)}$ to the direct sum of copies of the comparison map $K_n(\mathbb{C}) \rightarrow K_n^{\text{top}}(\mathbb{C})$. Since the latter map is not an isomorphism for $n \neq 0$, it follows that $\gamma_n^{\mathbb{C}}$ is not an isomorphism either.

Remark 9.10. It has been shown that if \mathfrak{A} is a properly infinite C^* -algebra then the comparison map $K_*(\mathfrak{A}) \rightarrow K_*^{\text{top}}(\mathfrak{A})$ is an isomorphism [8]. Thus $K_*(C_{\mathbb{C}}^*(E)) \rightarrow K_*^{\text{top}}(C_{\mathbb{C}}^*(E))$ is an isomorphism whenever $C_{\mathbb{C}}^*(E)$ is properly infinite.

The following proposition is a variant of a theorem of Higson (see [23, 3.4]) that asserts that stable C^* -algebras are K -regular.

Proposition 9.11. *Let A be an H' -unital ring, and \mathfrak{B} a stable C^* -algebra. Then $A \otimes \mathfrak{B}$ is K -regular.*

Proof. By Lemma 2.3 we may assume that A is a \mathbb{Q} -algebra. Since $A \rightarrow A[t]$ preserves H -unitality, the proposition amounts to showing that the functor $A \mapsto K_*(A \otimes \mathfrak{B})$ is invariant under polynomial homotopy. Observe that if \mathfrak{A} is any C^* -algebra, then $A \otimes (\mathfrak{B} \bar{\otimes} \mathfrak{A})$ is H -unital, which implies that the functor $A \mapsto E(A) = K_*(A \otimes (\mathfrak{B} \bar{\otimes} \mathfrak{A}))$, which is stable (because K -theory is matrix stable on H' -unital rings), is also split exact. Hence E is invariant under continuous homotopies, by Higson's homotopy invariance theorem [19]. Thus E sends all the evaluation maps $\text{ev}_t : \mathbb{C}[0, 1] \rightarrow \mathbb{C}$ to the same map. But since the evaluation maps $\text{ev}_i : A[t] \rightarrow A$ factor through $\text{ev}_i : A \otimes \mathbb{C}[0, 1] \rightarrow A$, it follows that $A \mapsto E(\mathbb{C}) = K_*(A \otimes \mathfrak{B})$ is invariant under polynomial homotopies, as we had to prove. \square

Corollary 9.12. *If \mathfrak{B} is a stable C^* -algebra and E a row-finite quiver, then both \mathfrak{B} and $L_{\mathfrak{B}}(E)$ are K -regular, and the map of Theorem 6.3*

$$\text{hocofiber}(K(\mathfrak{B})^{(E_0 \setminus \text{Sink}(E))} \xrightarrow{1-N_E^t} K(\mathfrak{B})^{(E_0)}) \rightarrow K(L_{\mathfrak{B}}(E))$$

is an equivalence.

Proof. That \mathfrak{B} and $L_{\mathfrak{B}}(E)$ are K -regular is immediate from the proposition; by Corollary 2.4, they are also H -unital. It follows from this and from Lemma 8.2 that the comparison maps $K(\mathfrak{B}) \rightarrow KH(\mathfrak{B})$ and $K(L_{\mathfrak{B}}(E)) \rightarrow KH(L_{\mathfrak{B}}(E))$ are equivalences. Now apply Theorem 8.6. \square

Theorem 9.13. *If \mathfrak{B} is a stable C^* -algebra then the map $\gamma_n^{\mathfrak{B}}$ is an isomorphism for every n and every row-finite quiver E .*

Proof. The theorem is immediate from Corollary 9.12, Theorem 9.1, and the fact (proved in [20] for $n \leq 0$ and in [28] for $n \geq 1$) that the map $K_n(\mathfrak{B}) \rightarrow K_n^{\text{top}}(\mathfrak{B})$ is an isomorphism for all n . \square

Remark 9.14. If \mathfrak{B} is stable, then $C_{\mathfrak{B}}^*(E)$ is stable, and thus the comparison map $K_*(C_{\mathfrak{B}}^*(E)) \rightarrow K_*^{\text{top}}(C_{\mathfrak{B}}^*(E))$ is an isomorphism. Moreover we also have $KH(C_{\mathfrak{B}}^*(E)) \cong K_*^{\text{top}}(C_{\mathfrak{B}}^*(E))$, by 9.11.

Acknowledgement. Part of the research for this article was carried out during visits of the third named author to the Centre de Recerca Matemàtica and the Departament de Matemàtiques of the Universitat Autònoma de Barcelona. He is indebted to these institutions for their hospitality.

REFERENCES

- [1] G. Abrams, G. Aranda Pino. *The Leavitt path algebra of a graph*. J. Algebra **293** (2005), 319–334.
- [2] P. Ara, M. Brustenga. *Module theory over Leavitt path algebras and K -theory*. Preprint 2009.
- [3] P. Ara, M.A. González-Barroso, K.R. Goodearl, E. Pardo. *Fractional skew monoid rings* J. Algebra **278** (2004) 104–126.
- [4] P. Ara, M.A. Moreno, E. Pardo. *Nonstable K -theory for graph algebras*. Algebras Represent. Theory **10** (2007) 157–178.
- [5] F. Bühler. *Vogel's notion of regularity for non-coherent rings*. arXiv:math/0612569v1 [math.KT].
- [6] G. Cortiñas. *The obstruction to excision in K -theory and in cyclic homology*. Invent. Math. **454** (2006) 143–173.
- [7] G. Cortiñas. *Algebraic vs. topological K -theory: a friendly match*. Preprint. Available at <http://mate.dm.uba.ar/~gcorti/friendly.pdf>.
- [8] G. Cortiñas, N.C. Phillips. *Algebraic K -theory and properly infinite C^* -algebras*. Preprint.
- [9] G. Cortiñas, C. Haesemeyer, M. Schlichting, C. Weibel. *Cyclic homology, cdh-cohomology and negative K -theory*. Ann. of Math. **167**, (2008) 549–573.
- [10] G. Cortiñas, A. Thom. *Bivariant algebraic K -theory*. J. reine angew. Math. **510**, 71–124.
- [11] J. Cuntz. *K -theory for certain C^* -algebras*. Ann. Math. (2) **113** (1981), no. 1, 181–197.
- [12] J. Cuntz. *K -theory for certain C^* -algebras II*. J. Operator Theory **5** (1981), no. 1, 101–108.
- [13] J. Cuntz. *A class of C^* -algebras and topological Markov chains II: reducible chains and the Ext-functor for C^* -algebras*. Invent. Math. **63** (1981), no. 1, 25–40.
- [14] J. Cuntz, W. Krieger. *A class of C^* -algebras and topological Markov chains*. Invent. Math. **56** (1980), no. 3, 251–268.
- [15] J. Cuntz, R. Meyer, J. Rosenberg. *Topological and bivariant K -theory*. Oberwolfach Seminars, 36. Birkhäuser Verlag, Basel, 2007.
- [16] E. Friedlander, M.E. Walker. *Comparing K -theories for complex varieties*. Amer. J. of Math. **128** (2001) 779–810.
- [17] S. M. Gersten. *K -theory of free rings*. Comm. in Algebra **1** (1974) 39–64.
- [18] D. Grayson. *The K -theory of semilinear endomorphisms*. J. Algebra **113** (1988), 358–372.
- [19] N. Higson. *Algebraic K -theory of C^* -algebras*. Adv. in Math. **67**, (1988) 1–40.
- [20] M. Karoubi. *K -théorie algébrique de certaines algèbres d'opérateurs*. Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978), pp. 254–290, Lecture Notes in Math., 725, Springer, Berlin, 1979.
- [21] W. G. Leavitt. *The module type of a ring*. Trans. Amer. Math. Soc. **103** (1962) 113–130.
- [22] I. Raeburn, W. Szymański. *Cuntz-Krieger algebras of infinite graphs and matrices* Trans. Amer. Math. Soc. **356** (2004) 39–59.
- [23] J. Rosenberg. *Comparison between algebraic and topological K -theory for Banach algebras and C^* -algebras*. In Handbook of K -Theory, Friedlander, Eric M.; Grayson, Daniel R. (Eds.). Springer-Verlag, New York, 2005.
- [24] A. H. Schofield, *Representations of Rings over Skew Fields*, LMS Lecture Notes Series 92, Cambridge Univ. Press, Cambridge, UK, 1985.
- [25] C. Smith. *Unpublished notes, 2008*.
- [26] A. Suslin. *Excision in the integral algebraic K -theory*. Proceedings of the Steklov Institute of Mathematics **208** (1995) 255–279.
- [27] A. Suslin. *Algebraic K -theory of fields*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 222–244, Amer. Math. Soc., Providence, RI, 1987.
- [28] A. Suslin, M. Wodzicki. *Excision in Algebraic K -Theory*. Ann. of Math. (2) **136** (1992), 51–122.
- [29] T. Vorst. *Localization of the K -theory of polynomial extensions*. Math. Annalen **244** (1979) 33–54.

- [30] F. Waldhausen. *Algebraic K-theory of generalized free products I*. Ann. of Math. (2) **108** (1978), 135–204.
- [31] C. Weibel. The K-book: An introduction to algebraic K-theory. Available at <http://www.math.rutgers.edu/~weibel/Kbook.html>.
- [32] C. Weibel. *Homotopy Algebraic K-theory*. Contemporary Math. **83** (1989) 461–488.
- [33] M. Wodzicki. *Excision in cyclic homology and in rational algebraic K-Theory*. Ann. of Math. (2) **129** (1989), 591–639.
- [34] D. Yao. *A note on the K-theory of twisted projective lines and twisted Laurent polynomial rings*. J. Algebra **123** (1995) 424–435.

P. Ara, M. Brustenga
 Departament de Matemàtiques
 Universitat Autònoma de Barcelona
 08193 Bellaterra (Barcelona), Spain
 E-mail: para@mat.uab.cat, mbrusten@mat.uab.cat

G. Cortiñas
 Dep. Matemática
 Ciudad Universitaria Pab 1
 1428 Buenos Aires, Argentina
 E-mail: gorti@dm.uba.ar
 URL: <http://mate.dm.uba.ar/~gorti>